

Low-energy limit of the extended Linear Sigma Model

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The extended Linear Sigma Model (eLSM) is an effective hadronic model based on the linear realization of chiral symmetry $SU(N_f)_L \times SU(N_f)_R$, with (pseudo)scalar and (axial-)vector mesons as degrees of freedom. In this paper, we study the low-energy limit of the eLSM for $N_f = 2$ flavors by integrating out all fields except for the pions, the (pseudo-)Nambu–Goldstone bosons of chiral symmetry breaking. We only keep terms entering at tree level and up to fourth order in powers of derivatives of the pion fields. Up to this order, there are four low-energy coupling constants in the resulting low-energy effective action. We show that the latter is formally identical to Chiral Perturbation Theory (ChPT), after choosing a representative for the coset space generated by chiral symmetry breaking and expanding up to fourth order in powers of derivatives of the pion fields. Two of the low-energy coupling constants of the eLSM are uniquely determined by a fit to meson masses and decay widths. We find that their tree-level values are in reasonable agreement with the corresponding low-energy coupling constants of ChPT. The other two low-energy coupling constants are functions of parameters that can in principle be determined by $\pi\pi$ scattering, which has not yet been studied within the eLSM. Therefore, we use the respective values from ChPT to make a prediction for the values of these parameters in the eLSM Lagrangian.

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I. INTRODUCTION

The physics of the strong interaction is described by Quantum Chromodynamics (QCD). For N_f massless quark flavors the classical QCD Lagrangian possesses a global $U(N_f)_L \times U(N_f)_R \cong U(N_f)_V \times U(N_f)_A$ symmetry. At quantum level, this symmetry is reduced to $SU(N_f)_V \times SU(N_f)_A \times U(1)_V$, since the $U(1)_A$ symmetry is explicitly broken by a quantum anomaly [1]. The $U(1)_V$ symmetry corresponds to quark number conservation. Since it is trivially fulfilled in any theory with hadrons as degrees of freedom, we do not need to consider it in the following. The remaining $SU(N_f)_V \times SU(N_f)_A$ symmetry, the so-called chiral symmetry, is explicitly broken to $SU(N_f)_V$ by nonvanishing and equal quark masses, and to the direct product of $N_f - 1$ separate $U(1)$ groups if all quark masses are unequal. It is well known that the experimentally observed hadrons can be grouped into irreducible representations of $SU(N_f)_V$ and not into those of $SU(N_f)_V \times SU(N_f)_A$ [2]. This observation provides strong evidence for the fact that chiral symmetry must be spontaneously broken to its diagonal flavor subgroup $SU(N_f)_V$. As a consequence of the spontaneous breakdown of chiral symmetry, we expect the occurrence of $N_f^2 - 1$ (pseudo-)Nambu–Goldstone bosons. Throughout this work, we restrict ourselves to the two-flavor case, $N_f = 2$. Then, the three (pseudo-)Nambu–Goldstone bosons are given by the pion isotriplet $\vec{\pi}$.

Another important property of QCD is that its running coupling constant α_S becomes large at small energies. This phenomenon implies that nonperturbative methods are needed to investigate the low-energy spectrum of QCD. Besides lattice methods, one can also use Effective Field Theories (EFTs) to investigate the low-energy dynamics of QCD. The most prominent, systematic, and well-defined approach of this type is Chiral Perturbation Theory (ChPT), see e.g. Refs. [3–7] and refs. therein. ChPT is a theory which describes the dynamics of the (pseudo)Nambu–Goldstone bosons, i.e., for $N_f = 2$ the pions. In ChPT, chiral symmetry is nonlinearly realized, i.e., the Nambu–Goldstone bosons enter as parameters of the representative of the coset space $SU(N_f) \times SU(N_f)/SU(N_f)$ of chiral symmetry breaking. ChPT is defined by a Lagrangian containing all chiral invariants constructed from powers of derivatives of the coset representative. The coupling constants multiplying these invariants are the so-called low-energy constants (LECs). Since the Lagrangian contains an arbitrary number of derivatives of the coset representative, it is not perturbatively renormalizable. However, a power series in derivatives of the coset representative is equivalent to a power series in $p/(4\pi f_\pi)$, where p is the momentum of the pion field and f_π the pion decay constant. Thus, for small pion momenta this power series is expected to converge. Moreover, one can remove all infinities order by order in the pion momentum by absorbing them into the LECs. The fundamentals of ChPT were investigated in Ref. [4] where it was shown that ChPT has the very same Green functions as QCD in the low-energy limit. In conclusion, ChPT is definitely the best approach that we have to study the interactions of (slow) pions.

ChPT was extended by including vector mesons, see e.g. Refs. [8, 9]. In the seminal work of Ref. [8] it was shown they play an important role in determining the values of the LECs. Yet, ChPT becomes less and less accurate when the energy scale increases. In particular, both the scalar (up to 1.7 GeV) and the axial-vector (up to 1.5 GeV) sectors are notoriously problematic due to the existence of broad resonances [such as $f_0(500)$, $f_0(1370)$, $a_1(1230)$] and of resonances close to thresholds in (pseudo-)Nambu–Goldstone boson scattering processes [i.e., $a_0(980)$ and $f_0(980)$]. The scalar sector is also important since it is related to both the chiral and the gluon condensate (i.e., the vacuum expectation values of the chiral partner of the pion and of the scalar dilaton/glueball field).

An alternative approach to the low-energy dynamics of QCD is given by hadronic models based on a linear realization of chiral symmetry. Such models are usually referred to as Linear Sigma Models (LSMs), which historically were studied even before ChPT [10] [see also Refs. [11–13]]. A significant difference between the two approaches is that in LSMs the chiral partners of the (pseudo-)Nambu–Goldstone bosons appear on an equal footing, i.e., for $N_f = 2$ the scalar sigma field σ_N enters besides the pseudoscalar pions. However, the simple LSM with just pions and sigma does not have the same low-energy limit as QCD, i.e., its LECs do not assume the same values as in ChPT [3]. The LSM was extended by (axial-)vector degrees of freedom in Refs. [14, 15]. More recently, the so-called extended Linear Sigma Model (eLSM) was developed, which contains all quark-antiquark mesons with (pseudo)scalar and (axial-)vector quantum numbers below 2 GeV in mass. The Lagrangian of the eLSM is constructed to respect the chiral and the dilatation symmetry of QCD and to reflect the pattern of their respective breaking in nature. Requiring dilatation symmetry and demanding that only positive semi-definite powers of the dilaton field enter the Lagrangian of the eLSM implies that the latter contains only a finite number of chiral invariants. The eLSM was first presented for $N_f = 2$ in Refs. [16, 17] and then enlarged to $N_f = 3$ in Refs. [18, 19]. It was also studied for $N_f = 4$ [20] and for baryons in the vacuum [21] and at nonzero density [22].

A fit of the parameters of the eLSM to experimentally measured masses and decay widths shows an agreement on the 5% level [18]. This is remarkable, given the simplicity of the assumptions underlying the eLSM. A natural question then arises: does the eLSM have the same low-energy limit as QCD, i.e., does it reproduce the LECs of ChPT? In order to answer this question, we proceed as follows. On the one hand we choose a definite representation for the coset representative and expand the ChPT Lagrangian up to fourth order in powers of derivatives of the pion field. Then, the coupling constants of the resulting Lagrangian are well-defined functions of the LECs of ChPT. On the other hand, we successively integrate out all fields of the eLSM except for the pions. In this paper, we work at tree-level, i.e., we neglect all loop corrections, and keep terms up to fourth order in the pion fields. This enables us to perform this integration in a completely analytical way. The resulting low-energy effective action has the same mathematical form as the above mentioned Lagrangian resulting from expanding ChPT in powers of derivatives of the pion field. However, the respective coupling constants are now well-defined functions of the parameters of the eLSM, most of which were previously determined by the fit of Ref. [18]. The question whether the eLSM has the same low-energy limit as QCD thus boils down to how well its low-energy coupling constants compare to the values obtained from ChPT. A positive answer would validate the eLSM as a low-energy model for QCD.

The paper is organized as follows: Sec. II includes a short summary of $N_f = 2$ ChPT. In Sec. III we briefly present the $N_f = 2$ version of the eLSM and show in detail how to derive its low-energy limit. In Sec. IV we compare the numerical results for the low-energy coupling constants in ChPT and the eLSM. Finally, in Sec. V we present our conclusions and an outlook for future studies. We defer lengthy formulas to App. A. Appendix B contains a discussion of other scenarios: the case without (axial-)vector mesons and the case in which the resonance $f_0(500)$, a putative four-quark state, is regarded as the chiral partner of the pion.

II. CHIRAL PERTURBATION THEORY

ChPT is a well-defined low-energy EFT of QCD. It relies on a systematic low-energy analysis of the hadronic n -point functions built from scalar, pseudoscalar, vector, and axial-vector quark bilinears. The structure of these n -point functions is determined by chiral symmetry, since they have to transform in some representation of $SU(N_f)_V \times SU(N_f)_A$. In addition to that, chiral symmetry gives rise to symmetry relations among these n -point functions, the so-called Ward-Fradkin-Takahashi (WFT) identities. These symmetry relations allow for a systematic analysis of the hadronic n -point functions.

Another important property of the hadronic n -point functions is that they always have a pole whenever an intermediate particle can be created on-shell. In the case of QCD, the pole with the smallest energy that one observes corresponds to an on-shell pion, which shows that the low-energy dynamics of QCD is determined by the interactions of the pions among themselves. The idea behind ChPT is to perform a so-called chiral expansion, i.e., a simultaneous expansion in powers of quark masses and pion momenta of the QCD generating functional (with external fields coupling to the above mentioned quark bilinears)

$$Z_{QCD} = Z_2 + Z_4 + \dots, \quad (1)$$

where the different terms in this expansion Z_{2n} , $n = 1, 2, \dots$, include all possible combinations of the coset representative of chiral symmetry breaking and the external fields which are allowed by local chiral symmetry as well as by CPT and proper orthochronous Lorentz transformations. It is therefore clear that the number of allowed interaction terms rapidly increases with the order of the expansion. Up to and including next-to-leading order (NLO), the most general chiral Lagrangian is given by

$$\mathcal{L}_{\chi PT} = \mathcal{L}_2 + \mathcal{L}_4, \quad (2)$$

where the leading-order (LO) and NLO terms of the Lagrangian are given by Eqs. (A1) and (A2) in App. A 1. At LO the chiral Lagrangian contains only two free parameters, the pion decay constant f_π and the constant B_0 , which is related to the bare quark mass. At NLO the number of free parameters increases to ten. In this case, one has seven LECs ℓ_i , $i = 1, \dots, 7$, and three additional coupling constants h_i , $i = 1, 2, 3$, see Eq. (A2).

In this work, we are interested in the detailed interaction structure of the pion fields among themselves. Therefore, we choose

$$\mathcal{U} = \frac{1}{f_\pi} (\sigma + i\pi_i \tau^i) \quad \text{with } \sigma = f_\pi \sqrt{1 - \pi_i^2/f_\pi^2} \quad (3)$$

as a parametrization of the coset space $SU(2) \times SU(2)/SU(2)$ and expand the chiral Lagrangian, only keeping terms with up to four pion fields and space-time derivatives. The resulting Lagrangian takes the form

$$\begin{aligned} \mathcal{L}_{\chi PT} = & \frac{1}{2} (\partial_\mu \vec{\pi})^2 - \frac{1}{2} M_\pi^2 \vec{\pi}^2 + C_{1,\chi PT} (\vec{\pi}^2)^2 + C_{2,\chi PT} (\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 + C_{3,\chi PT} (\partial_\mu \vec{\pi})^2 (\partial_\nu \vec{\pi})^2 \\ & + C_{4,\chi PT} [(\partial_\mu \vec{\pi}) \cdot \partial_\nu \vec{\pi}]^2 + \mathcal{O}(\pi^6, \partial^6), \end{aligned} \quad (4)$$

where

$$M_\pi^2 = M^2 + \frac{2\ell_3}{f_\pi^2} M^4 \quad (5)$$

defines the NLO tree-level mass of the pion. The low-energy coupling constants $C_{i,\chi PT}$, $i = 1, \dots, 4$, are given by

$$C_{1,\chi PT} = -\frac{M^2}{8f_\pi^2}, \quad (6)$$

$$C_{2,\chi PT} = \frac{1}{2f_\pi^2}, \quad (7)$$

$$C_{3,\chi PT} = \frac{\ell_1}{f_\pi^4}, \quad (8)$$

$$C_{4,\chi PT} = \frac{\ell_2}{f_\pi^4}, \quad (9)$$

where the LECs are defined in Eq. (A2).

III. LOW-ENERGY LIMIT OF THE ELSM

A. Mesonic part of the eLSM

The eLSM is a linear sigma model which contains, besides the standard scalar and pseudoscalar mesons, also vector and axial-vector mesons [16–19]. All mesonic fields are interpreted as quark-antiquark states, such as e.g. found in the relativistic quark model of Ref. [23]. This identification is confirmed by the study of the large- N_c behavior [24] of masses and widths, as discussed in Ref. [18].

Scalar and pseudoscalar mesons are described by the matrix

$$\Phi = (\sigma_N + i\eta_N)T^0 + (\vec{a}_0 + i\vec{\pi}) \cdot \vec{T}, \quad (10)$$

where $T^0 = \mathbb{1}_{2 \times 2}/2$ and $\vec{T} = \vec{\tau}/2$, in which $\vec{\tau}$ denotes the vector of the Pauli matrices. The quantity $\vec{\pi}$ describes the pion triplet, while η_N describes the non-strange content of the η and η' mesons. Furthermore, the scalar triplet \vec{a}_0 is identified with $a_0(1450)$ [the alternative identification with $a_0(980)$ turns out to be unfavored [16, 18]]. Similarly,

the scalar isosinglet σ_N corresponds to the resonance $f_0(1370)$ [also in this case, the assignment to the light $f_0(500)$ is unfavorable, see App. B 2 for further discussion].

Vector and axial-vector mesons are described by the left- and right-handed fields

$$L^\mu = (\omega_N^\mu + f_{1N}^\mu)T^0 + (\vec{\rho}^\mu + \vec{a}_1^\mu) \cdot \vec{T} , \quad (11)$$

$$R^\mu = (\omega_N^\mu - f_{1N}^\mu)T^0 + (\vec{\rho}^\mu - \vec{a}_1^\mu) \cdot \vec{T} , \quad (12)$$

where the vector and axial-vector singlets ω_N^μ and f_{1N}^μ represent the $\omega(782)$ and $f_1(1285)$ mesons, respectively. In the isotriplet sector, $\vec{\rho}^\mu$ represents the vector meson $\rho(770)$ and \vec{a}_1^μ the axial-vector meson $a_1(1260)$.

The fields (10), (11), and (12) have a well-defined transformation behavior with respect to $U(2)_L \times U(2)_R$ transformations:

$$\Phi \xrightarrow{U(2)_L \times U(2)_R} U_L \Phi U_R^\dagger , \quad L^\mu \xrightarrow{U(2)_L \times U(2)_R} U_L L^\mu U_L^\dagger , \quad R^\mu \xrightarrow{U(2)_L \times U(2)_R} U_R R^\mu U_R^\dagger . \quad (13)$$

The most general chirally symmetric Lagrangian which contains operators of dimension (up to) four and reproduces the chiral symmetry breaking pattern found in nature is given by:

$$\begin{aligned} \mathcal{L}_{eLSM} = & \text{Tr} \left\{ (D^\mu \Phi)^\dagger D_\mu \Phi \right\} - m_0^2 \text{Tr} \left\{ \Phi^\dagger \Phi \right\} - \lambda_1 \left(\text{Tr} \left\{ \Phi^\dagger \Phi \right\} \right)^2 - \lambda_2 \text{Tr} \left\{ (\Phi^\dagger \Phi)^2 \right\} \\ & - \frac{1}{4} \text{Tr} \left\{ L^{\mu\nu} L_{\mu\nu} + R^{\mu\nu} R_{\mu\nu} \right\} + \frac{m_1^2}{2} \text{Tr} \left\{ L^\mu L_\mu + R^\mu R_\mu \right\} + \text{Tr} \left\{ H (\Phi^\dagger + \Phi) \right\} \\ & + c_1 \left(\det \Phi - \det \Phi^\dagger \right)^2 + i \frac{g_2}{2} \left(\text{Tr} \left\{ L^{\mu\nu} [L_\mu, L_\nu]_- \right\} + \text{Tr} \left\{ R^{\mu\nu} [R_\mu, R_\nu]_- \right\} \right) \\ & + \frac{h_1}{2} \text{Tr} \left\{ \Phi^\dagger \Phi \right\} \text{Tr} \left\{ L^\mu L_\mu + R^\mu R_\mu \right\} + h_2 \text{Tr} \left\{ |L^\mu \Phi|^2 + |\Phi R^\mu|^2 \right\} + 2h_3 \text{Tr} \left\{ \Phi R^\mu \Phi^\dagger L_\mu \right\} \\ & + g_3 \left(\text{Tr} \left\{ L^\mu L^\nu L_\mu L_\nu \right\} + \text{Tr} \left\{ R^\mu R^\nu R_\mu R_\nu \right\} \right) + g_4 \left(\text{Tr} \left\{ L^\mu L_\mu L^\nu L_\nu \right\} + \text{Tr} \left\{ R^\mu R_\mu R^\nu R_\nu \right\} \right) \\ & + g_5 \text{Tr} \left\{ L^\mu L_\mu \right\} \text{Tr} \left\{ R^\mu R_\mu \right\} + g_6 \left(\text{Tr} \left\{ L^\mu L_\mu \right\} \text{Tr} \left\{ L^\nu L_\nu \right\} + \text{Tr} \left\{ R^\mu R_\mu \right\} \text{Tr} \left\{ R^\nu R_\nu \right\} \right) , \end{aligned} \quad (14)$$

where $D_\mu \Phi = \partial_\mu \Phi - ig_1 (L^\mu \Phi - \Phi R^\mu)$, $H = h_{N,0} T^0$, and $h_{N,0} \sim m_u = m_d$.

Explicit breaking of chiral symmetry due to non-vanishing quark masses is affected by the term

$$\text{Tr} \left\{ H (\Phi^\dagger + \Phi) \right\} = h_{N,0} \sigma_N \quad (15)$$

which tilts the potential into the σ_N -direction. The $U(1)_A$ anomaly is incorporated via the term

$$c_1 \left(\det \Phi - \det \Phi^\dagger \right)^2 . \quad (16)$$

Spontaneous breaking of chiral symmetry is induced by a non-vanishing vacuum expectation value $\phi_N \equiv \langle \sigma_N \rangle$ of the σ_N field. Physical excitations of this field, corresponding to the σ_N meson, are described by performing a shift, $\sigma_N \longrightarrow \phi_N + \sigma_N$, in the Lagrangian (14). One then obtains the tree-level masses of the different mesons from terms quadratic in the fields. In addition, this shift leads to bilinear terms which mix the axial-vector and pseudoscalar fields, respectively. By shifting the axial-vector fields in an appropriate way,

$$f_{1N}^\mu \longrightarrow f_{1N}^\mu + Z w \partial^\mu \eta_N , \quad \vec{a}_1^\mu \longrightarrow \vec{a}_1^\mu + Z w \partial^\mu \vec{\pi} , \quad (17)$$

with

$$w \equiv \frac{g_1 \phi_N}{m_{a_1}^2} , \quad Z \equiv (1 - g_1 \phi_N w)^{-\frac{1}{2}} , \quad (18)$$

the bilinear terms in the Lagrangian can be eliminated [for details, see Refs. [18, 19]]. In addition, we have to redefine the pseudoscalar fields,

$$\eta_N \longrightarrow Z \eta_N , \quad \vec{\pi} \longrightarrow Z \vec{\pi} , \quad (19)$$

in order to obtain canonically normalized fields. In the end, the tree-level masses of the mesons read:

$$\begin{aligned}
m_\pi^2 &= \left[-m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2} \right) \phi_N^2 \right] Z^2, \\
m_{\eta_N}^2 &= \left[-m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2} \right) \phi_N^2 + c_1 \phi_N^2 \right] Z^2, \\
m_{a_0}^2 &= -m_0^2 + \left(\lambda_1 + \frac{3\lambda_2}{2} \right) \phi_N^2, \\
m_{\sigma_N}^2 &= -m_0^2 + 3 \left(\lambda_1 + \frac{\lambda_2}{2} \right) \phi_N^2, \\
m_{\omega_N}^2 &= m_\rho^2 = m_1^2 + \frac{1}{2} (h_1 + h_2 + h_3) \phi_N^2, \\
m_{f_{1N}}^2 &= m_{a_1}^2 = m_1^2 + \frac{1}{2} (h_1 + h_2 - h_3) \phi_N^2 + g_1^2 \phi_N^2.
\end{aligned} \tag{20}$$

The parameters m_0^2 , m_1^2 , λ_2 , g_1 , g_2 , h_2 , h_3 , $h_{N,0}$, and c_1 of the Lagrangian (14) were determined in Ref. [18] through a fit of the tree-level masses (20) as well as several decay widths to experimental data. The large- N_c suppressed parameters λ_1 and h_1 only influence properties of the scalar-isoscalar mesons and were excluded from this fit. Setting them to zero, the fit allows predictions for the masses of these mesons. They turn out to be in the range of masses of the experimentally observed scalar-isoscalar states $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ [18]. Note that the fit of Ref. [18] was performed for the $N_f = 3$ version of the eLSM, while in this work we consider $N_f = 2$. Nevertheless, when $\lambda_1 = h_1 = 0$, the terms distinguishing between the cases $N_f = 2$ and $N_f = 3$ in Eqs. (20) vanish, such that these relations for the masses hold with the identification $c_1^{(N_f=2)} = c_1^{(N_f=3)} \phi_S^2$, where ϕ_S is the strange quark condensate.

The coupling constants g_3 , g_4 , g_5 , g_6 in the Lagrangian (14) describe four-point interactions between vector mesons. They do not enter masses and decay widths of mesons at tree-level and thus were not determined in Ref. [18]. As we shall see in Sec. IIIB, they influence the values of the low-energy coupling constants because of the mixing of axial-vector and pseudoscalar mesons, Eq. (17), which gives rise to a four-pion term. While the constants g_5 and g_6 can be dismissed in virtue of large- N_c considerations, the constants g_3 and g_4 are not expected to be small.

B. Determination of the low-energy effective action of the eLSM

In this subsection, we compute the low-energy effective action of the eLSM by successively integrating out all fields except for the pions in the vacuum-to-vacuum transition amplitude $\langle f, \infty | f, -\infty \rangle$, where $f = \{\sigma_N, \eta_N, \vec{a}_0, \vec{\pi}, \omega_N^\mu, f_{1N}^\mu, \vec{\rho}^\mu, \vec{a}_1^\mu\}$. This transition amplitude can be written as a functional integral over all fields

$$\langle f, \infty | f, -\infty \rangle = \mathcal{N} \int \mathcal{D}f \exp \left(i \int d^4x \mathcal{L}_{eLSM} \right), \tag{21}$$

where \mathcal{N} is a normalization constant. Our aim is to obtain a Lagrangian which contains only pions and can then be compared to ChPT. Hence, we integrate out the heavy mesonic fields $H \equiv \{\sigma_N, \eta_N, \vec{a}_0, \omega_N^\mu, f_{1N}^\mu, \vec{\rho}^\mu, \vec{a}_1^\mu\}$. In general, this is a formidable task, since the various interaction structures couple the functional integrations over different fields in Eq. (21). Furthermore, there are cubic and quartic (self-)interaction terms of the heavy fields, which prevent a straightforward analytic solution of the respective functional integral.

Nevertheless, if we restrict the comparison of ChPT with the low-energy effective action of the eLSM to *tree-level* and to *fourth order in powers of derivatives of the pion fields*, it is possible to make progress by purely analytical means. We first observe that the Lagrangian (14) contains the following type of interaction terms:

- (1) terms containing three or four heavy fields, but no pion field, $\Gamma_H^{(3)} \equiv H_i H_j H_k$ and $\Gamma_H^{(4)} \equiv H_i H_j H_k H_l$, where H_i, H_j, H_k, H_l are heavy fields,
- (2) terms containing one pion and two or three heavy fields, $\Gamma_\pi^{(3)} \equiv H_i H_j \pi$ and $\Gamma_\pi^{(4)} \equiv H_i H_j H_k \pi$,
- (3) terms containing two pion fields and one heavy field, $\Gamma_{\pi\pi}^{(3)} \equiv H_i \pi \pi$,
- (4) terms containing two pion fields and two heavy fields, $\Gamma_{\pi\pi}^{(4)} \equiv H_i H_j \pi \pi$,
- (5) terms containing three pion fields and one heavy field $\Gamma_{\pi\pi\pi}^{(4)} \equiv H_i \pi \pi \pi$, and

(6) terms containing four pion fields $\Gamma_{\pi\pi\pi\pi}^{(4)} \equiv \pi\pi\pi\pi$.

Under the above assumptions, we may now neglect all terms except those of type (3) and (6). This can be proved as follows. Because of our assumption to consider only terms up to fourth order in powers of derivatives of the pion fields, we need to combine the different types of vertices in a way which generates four-pion interaction terms when integrating out the heavy fields. However, one can convince oneself via a simple graphical analysis that most of these combinations then contain loops of heavy fields. By our assumption to consider only tree-level contributions to the low-energy effective action, these can therefore be neglected. The only way to generate a tree-level contribution is to combine two vertices of type (3) with the same heavy field H_i . When integrating out the latter, this generates a diagram where the two vertices of type (3) are connected by a propagator for the field H_i . Other than that, the only other terms that contribute at tree-level are those of type (6).

After these considerations, the only terms of relevance in the Lagrangian (14) are

$$\mathcal{L}_{eLSM} = \mathcal{L}_\pi + \mathcal{L}_{\sigma_N\pi} + \mathcal{L}_{\rho\pi} + \dots, \quad (22)$$

where

$$\begin{aligned} \mathcal{L}_\pi = & \frac{1}{2} (\partial_\mu \vec{\pi})^2 - \frac{1}{2} m_\pi^2 \vec{\pi}^2 + \frac{g_1^2}{2} w^2 Z^4 (\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 - \frac{1}{4} \left(\lambda_1 + \frac{\lambda_2}{2} \right) Z^4 (\vec{\pi}^2)^2 + \frac{1}{4} (h_1 + h_2 - h_3) w^2 Z^4 \vec{\pi}^2 (\partial_\mu \vec{\pi})^2 \\ & + \frac{h_3}{2} w^2 Z^4 (\vec{\pi} \times \partial_\mu \vec{\pi})^2 + \left(-\frac{g_3}{4} + \frac{g_4}{4} + \frac{g_5}{4} + \frac{g_6}{2} \right) w^4 Z^4 (\partial_\mu \vec{\pi})^2 (\partial_\nu \vec{\pi})^2 + \frac{g_3}{2} w^4 Z^4 (\partial^\mu \vec{\pi}) \cdot (\partial^\nu \vec{\pi}) (\partial_\mu \vec{\pi}) \cdot \partial_\nu \vec{\pi} \end{aligned} \quad (23)$$

contains the kinetic and mass contributions of the π -field as well as all types of four-pion interaction terms, and

$$\begin{aligned} \mathcal{L}_{\sigma_N\pi} = & \frac{1}{2} (\partial_\mu \sigma_N)^2 - \frac{1}{2} m_{\sigma_N}^2 \sigma_N^2 - \left(\lambda_1 + \frac{\lambda_2}{2} \right) \phi_N Z^2 \sigma_N \vec{\pi}^2 + g_1 w Z^2 (\partial^\mu \sigma_N) (\partial_\mu \vec{\pi}) \cdot \vec{\pi} \\ & + \left\{ \left[g_1^2 \phi_N + (h_1 + h_2 - h_3) \frac{\phi_N}{2} \right] w^2 Z^2 - g_1 w Z^2 \right\} \sigma_N (\partial_\mu \vec{\pi})^2, \end{aligned} \quad (24)$$

as well as

$$\begin{aligned} \mathcal{L}_{\rho\pi} = & -\frac{1}{4} \vec{\rho}^{\mu\nu} \cdot \vec{\rho}_{\mu\nu} + \frac{1}{2} m_\rho^2 \vec{\rho}_\mu^2 - \frac{\xi_\rho}{2} (\partial_\mu \vec{\rho}^\mu)^2 + g_2 w^2 Z^2 (\partial^\mu \vec{\rho}^\nu) \cdot (\partial_\nu \vec{\pi} \times \partial_\mu \vec{\pi}) \\ & + \left[(g_1^2 \phi_N - h_3 \phi_N) w Z^2 - g_1 Z^2 \right] \vec{\rho}^\mu \cdot (\partial_\mu \vec{\pi} \times \vec{\pi}) \end{aligned} \quad (25)$$

contain the kinetic and mass terms as well as the terms linear in the σ_N and the ρ field, respectively. Note that we added a Stückelberg term $\xi_\rho (\partial_\mu \vec{\rho}^\mu)^2 / 2$ to the Lagrangian in order to make the inverse ρ -meson propagator invertible, cf. App. A 2.

The remaining terms [denoted by the ellipsis in Eq. (22)] contain the kinetic and mass terms for the other heavy fields $H = \eta_N, \vec{a}_0, \omega_N^\mu, f_{1N}^\mu, \vec{a}_1^\mu$ as well as their interaction terms with pions. However, they do not contain any term of type (3), cf. App. A 2, and therefore can be neglected within our approximation scheme.

Since the Lagrangian (22) contains at most quadratic terms in the heavy fields, the integration over the latter in the functional integral (21) is of (shifted) Gaussian type and can therefore be performed analytically. Using Eq. (22) the functional integral (21) takes the form:

$$\langle f, \infty | f, -\infty \rangle = \mathcal{N} \int \mathcal{D}\pi \exp \left(i \int d^4x \mathcal{L}_\pi \right) \prod_{H=\sigma_N, \rho} I_H[\pi], \quad (26)$$

where

$$I_{\sigma_N}[\pi] = \int \mathcal{D}\sigma_N \exp \left(i \int d^4x \mathcal{L}_{\sigma_N\pi} \right), \quad I_\rho[\pi] = \int \mathcal{D}\sigma_N \exp \left(i \int d^4x \mathcal{L}_{\rho\pi} \right), \quad (27)$$

with $\mathcal{L}_{\sigma_N\pi}$ and $\mathcal{L}_{\rho\pi}$ are given by Eqs. (24) and (25), respectively. Since both functional integrals decouple, the order in which we integrate out these fields is irrelevant. We start with σ_N :

$$\begin{aligned} I_{\sigma_N}[\pi] = & \int \mathcal{D}\sigma_N \exp \left[-\frac{i}{2} \int d^4x d^4y \sigma_N(x) \mathcal{O}_{\sigma_N}(x, y) \sigma_N(y) + i \int d^4x J_{\sigma_N\pi}(x) \sigma_N(x) \right] \\ = & \mathcal{N}_{\sigma_N} \exp \left[\frac{i}{2} \int d^4x d^4y J_{\sigma_N\pi}(x) \mathcal{O}_{\sigma_N}^{-1}(x, y) J_{\sigma_N\pi}(y) \right], \end{aligned} \quad (28)$$

where

$$\mathcal{O}_{\sigma_N}^{-1}(x, y) = (\square_x + m_{\sigma_N}^2)^{-1} \delta^{(4)}(x - y) = \frac{1}{m_{\sigma_N}^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{\square_x}{m_{\sigma_N}^2} \right)^n \delta^{(4)}(x - y) \quad (29)$$

and

$$J_{\sigma_N \pi}(x) = c_{1, \sigma_N} \vec{\pi}^2 + c_{2, \sigma_N} (\partial_\mu \vec{\pi})^2, \quad (30)$$

with the coefficients

$$c_{1, \sigma_N} = g_1 w Z^2 m_\pi^2 - \left(\lambda_1 + \frac{\lambda_2}{2} \right) \phi_N Z^2, \quad (31)$$

$$c_{2, \sigma_N} = \left[g_1^2 \phi_N + (h_1 + h_2 - h_3) \frac{\phi_N}{2} \right] w^2 Z^2 - 2g_1 w Z^2. \quad (32)$$

Expanding the sum in Eq. (29) to order $n = 2$, neglecting terms of higher than fourth order in derivatives of the pion fields, and using the equation of motion of the free pion field, Eq. (28) can be written as

$$\begin{aligned} I_{\sigma_N}[\pi] = \mathcal{N}_{\sigma_N} \exp \left(i \int d^4 x \left\{ \left[\frac{c_{1, \sigma_N}^2}{2m_{\sigma_N}^2} \left(1 - \frac{4m_\pi^4}{m_{\sigma_N}^4} \right) + \frac{c_{1, \sigma_N} c_{2, \sigma_N} m_\pi^2}{m_{\sigma_N}^2} \left(1 + \frac{2m_\pi^2}{m_{\sigma_N}^2} \right) \right] (\vec{\pi}^2)^2 \right. \right. \\ + \left[\frac{2c_{1, \sigma_N}^2}{m_{\sigma_N}^4} \left(1 + \frac{4m_\pi^2}{m_{\sigma_N}^2} \right) - \frac{2c_{1, \sigma_N} c_{2, \sigma_N}}{m_{\sigma_N}^2} \left(1 + \frac{2m_\pi^2}{m_{\sigma_N}^2} \right) \right] (\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 \\ \left. \left. + \left[\frac{c_{2, \sigma_N}^2}{2m_{\sigma_N}^2} - \frac{2c_{1, \sigma_N} c_{2, \sigma_N}}{m_{\sigma_N}^4} + \frac{2c_{1, \sigma_N}^2}{m_{\sigma_N}^6} \right] (\partial_\mu \vec{\pi})^2 (\partial_\nu \vec{\pi})^2 \right\} \right). \end{aligned} \quad (33)$$

We now turn to the contribution of the ρ -meson:

$$\begin{aligned} I_\rho[\pi] &= \int \mathcal{D}\rho \exp \left[\frac{1}{2} \int d^4 x d^4 y \vec{\rho}_\mu(x) \cdot \mathcal{O}_\rho^{\mu\nu}(x, y) \vec{\rho}_\nu(y) + i \int d^4 x \vec{J}_{\rho\pi}^\mu(x) \cdot \vec{\rho}_\mu(x) \right] \\ &= \mathcal{N}_\rho \exp \left[-\frac{i}{2} \int d^4 x d^4 y \vec{J}_{\rho\pi, \mu}(x) \cdot \mathcal{O}_\rho^{\mu\nu, -1}(x, y) \vec{J}_{\rho\pi, \nu}(y) \right], \end{aligned} \quad (34)$$

where

$$\mathcal{O}_\rho^{\mu\nu, -1}(x, y) = \left[\frac{g^{\mu\nu}}{\square_x + m_\rho^2} - \frac{1 - \xi_\rho}{(\xi_\rho \square_x + m_\rho^2)(\square_x + m_\rho^2)} \partial_x^\mu \partial_x^\nu \right] \delta^{(4)}(x - y)$$

is the propagator of the ρ -meson and

$$\vec{J}_{\rho\pi}^\mu(x) = c_{1, \rho} [(\partial^\mu \vec{\pi}) \times \vec{\pi}] - c_{2, \rho} [(\partial^\mu \partial^\nu \vec{\pi}) \times \partial_\nu \vec{\pi}], \quad (35)$$

with the coefficients

$$c_{1, \rho} = (g_1^2 - h_3) \phi_N w Z^2 - g_1 Z^2 + g_2 w^2 Z^2 m_\pi^2, \quad (36)$$

$$c_{2, \rho} = g_2 w^2 Z^2. \quad (37)$$

At this point, for the sake of simplicity we choose $\xi_\rho = 1$, which eliminates the term proportional to $\partial_x^\mu \partial_x^\nu$ in the inverse propagator. This does not influence our results, as one can show that this term results in four-pion interaction terms with six or more space-time derivatives, which we neglect in our treatment. Then, the inverse operator in Eq. (34) simplifies and the functional integral with respect to ρ^μ can finally be written as

$$\begin{aligned} I_\rho[\pi] = \mathcal{N}_\rho \exp \left(-i \int d^4 x \left\{ \left(\frac{c_{1, \rho}^2 m_\pi^2}{2m_\rho^2} - \frac{c_{1, \rho} c_{2, \rho} m_\pi^4}{m_\rho^2} \right) (\vec{\pi}^2)^2 - \left(\frac{3c_{1, \rho}^2}{2m_\rho^2} - \frac{3c_{1, \rho} c_{2, \rho} m_\pi^2}{m_\rho^2} \right) (\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 \right. \right. \\ \left. \left. + \left(\frac{c_{1, \rho}^2}{m_\rho^4} + \frac{c_{1, \rho} c_{2, \rho}}{m_\rho^2} \right) (\partial_\mu \vec{\pi})^2 (\partial_\nu \vec{\pi})^2 - \left(\frac{c_{1, \rho}^2}{m_\rho^4} + \frac{c_{1, \rho} c_{2, \rho}}{m_\rho^2} \right) [(\partial_\mu \vec{\pi}) \cdot \partial_\nu \vec{\pi}]^2 \right\} \right). \end{aligned} \quad (38)$$

In order to obtain the tree-level effective action of the eLSM, we have to insert Eqs. (33) and (38) into Eq. (26). The resulting tree-level effective Lagrangian has then exactly the same form as Eq. (4) obtained from ChPT:

$$\begin{aligned} \mathcal{L}_{eLSM,eff}[\vec{\pi}] = & \frac{1}{2} (\partial_\mu \vec{\pi})^2 - \frac{1}{2} m_\pi^2 \vec{\pi}^2 + C_{1,eLSM} (\vec{\pi}^2)^2 + C_{2,eLSM} [(\partial_\mu \vec{\pi}) \cdot \vec{\pi}]^2 + C_{3,eLSM} (\partial_\mu \vec{\pi})^2 (\partial_\nu \vec{\pi})^2 \\ & + C_{4,eLSM} [(\partial_\mu \vec{\pi}) \cdot \partial_\nu \vec{\pi}]^2, \end{aligned} \quad (39)$$

where the low-energy coupling constants $C_{i,eLSM}$, $i = 1, \dots, 4$, are functions of the parameters of the eLSM Lagrangian (14):

$$\begin{aligned} C_{1,eLSM} = & \frac{Z^4}{4} \left[(h_1 + h_2 + h_3) w^2 m_\pi^2 - \left(\lambda_1 + \frac{\lambda_2}{2} \right) \right] + \frac{c_{1,\sigma_N}^2}{2m_{\sigma_N}^2} \left(1 - \frac{4m_\pi^4}{m_{\sigma_N}^4} \right) \\ & + \frac{c_{1,\sigma_N} c_{2,\sigma_N} m_\pi^2}{m_{\sigma_N}^2} \left(1 + \frac{2m_\pi^2}{m_{\sigma_N}^2} \right) - \frac{c_{1,\rho}^2 m_\pi^2}{2m_\rho^2} + \frac{c_{1,\rho} c_{2,\rho} m_\pi^4}{m_\rho^2}, \end{aligned} \quad (40)$$

$$\begin{aligned} C_{2,eLSM} = & \frac{1}{2} (g_1^2 - h_1 - h_2 - 2h_3) w^2 Z^4 + \frac{2c_{1,\sigma_N}^2}{m_{\sigma_N}^4} \left(1 + \frac{4m_\pi^2}{m_{\sigma_N}^2} \right) - \frac{2c_{1,\sigma_N} c_{2,\sigma_N}}{m_{\sigma_N}^2} \left(1 + \frac{2m_\pi^2}{m_{\sigma_N}^2} \right) \\ & + \frac{3c_{1,\rho}^2}{2m_\rho^2} - \frac{3c_{1,\rho} c_{2,\rho} m_\pi^2}{m_\rho^2}, \end{aligned} \quad (41)$$

$$C_{3,eLSM} = \frac{1}{4} (-g_3 + g_4 + g_5 + 2g_6) w^4 Z^4 + \frac{c_{2,\sigma_N}^2}{2m_{\sigma_N}^2} - \frac{2c_{1,\sigma_N} c_{2,\sigma_N}}{m_{\sigma_N}^4} + \frac{2c_{1,\sigma_N}^2}{m_{\sigma_N}^6} - \frac{c_{1,\rho}^2}{m_\rho^4} - \frac{c_{1,\rho} c_{2,\rho}}{m_\rho^2}, \quad (42)$$

$$C_{4,eLSM} = \frac{g_3}{2} w^4 Z^4 + \frac{c_{1,\rho}^2}{m_\rho^4} + \frac{c_{1,\rho} c_{2,\rho}}{m_\rho^2}. \quad (43)$$

These expressions represent the main result of this paper. Note that they already contain contributions from the σ_N - and the ρ -meson at tree-level. In other approaches without these meson degrees of freedom, e.g. ChPT, such contributions only enter at higher-loop order.

IV. RESULTS

In this section we determine the numerical values for the low-energy coupling constants $C_{i,\chi PT}$ and $C_{i,eLSM}$, $i = 1, \dots, 4$, and compare them with each other.

A. ChPT

In ChPT at NLO the LECs are functions of the energy scale μ and are denoted as $\ell_i(\mu)$. They are related to the usually quoted μ -independent quantities $\bar{\ell}_i$ through the equation:

$$\bar{\ell}_i = \frac{32\pi^2}{\gamma_i} \ell_i(\mu) - \ln \frac{M_\pi^2}{\mu^2}, \quad (44)$$

where $\gamma_1 = 1/3$, $\gamma_2 = 2/3$, and $\gamma_3 = -1/2$, see Ref. [3]. The numerical values of the $\bar{\ell}_i$ are given in Ref. [25]. For the pion mass we use $M_\pi = (139.57018 \pm 0.00035)$ MeV and for the renormalization scale $\mu = 770$ MeV, resulting in the following values for the $\ell_i(\mu = 770 \text{ MeV})$:

$$\ell_1 = (-4.03 \pm 0.63) \cdot 10^{-3}, \quad (45)$$

$$\ell_2 = (1.87 \pm 0.21) \cdot 10^{-3}, \quad (46)$$

$$\ell_3 = (0.8 \pm 3.9) \cdot 10^{-3}. \quad (47)$$

In this way the mass parameter M , which is defined by the NLO tree-level mass of the pion, Eq. (5), reads:

$$M = (139.3 \pm 1.2) \text{ MeV}, \quad (48)$$

Upon using $f_\pi = (92.2 \pm 0.1)$ MeV [2] one obtains:

$$C_{1,\chi PT} = -0.29 \pm 0.34 , \quad (49)$$

$$C_{2,\chi PT} = (5.882 \pm 0.013) \cdot 10^{-5} \text{ MeV}^{-2} , \quad (50)$$

$$C_{3,\chi PT} = (-5.57 \pm 0.88) \cdot 10^{-11} \text{ MeV}^{-4} , \quad (51)$$

$$C_{4,\chi PT} = (2.58 \pm 0.29) \cdot 10^{-11} \text{ MeV}^{-4} . \quad (52)$$

B. eLSM

The parameters of the eLSM were determined in Ref. [18] through a fit to experimental data. We only quote those of relevance for the following:

$$g_1 = 5.843 \pm 0.018 , \quad (53)$$

$$g_2 = 3.02 \pm 0.23 , \quad (54)$$

$$h_2 = 9.88 \pm 0.66 , \quad (55)$$

$$h_3 = 4.87 \pm 0.086 \quad (56)$$

$$\lambda_2 = 68.297 \pm 0.044 , \quad (57)$$

$$m_0^2 = (-0.91825 \pm 0.00064) \text{ GeV}^2 \quad (58)$$

$$m_1^2 = (0.4135 \pm 0.015) \text{ GeV}^2 . \quad (59)$$

We also set $h_1 = \lambda_1 = g_5 = g_6 = 0$, which are suppressed in the large- N_c limit [24]. The minimum of the potential is at $\phi_N = (164.6 \pm 0.1)$ MeV. As a consequence, we obtain

$$m_\pi = (141.0 \pm 5.8) \text{ MeV} , \quad (60)$$

$$m_{\sigma_N} = 1362.7 \text{ MeV} , \quad (61)$$

$$m_\rho = (783.1 \pm 7.0) \text{ MeV} , \quad (62)$$

$$m_{a_1} = (1185.6 \pm 5.6) \text{ MeV} , \quad (63)$$

$$w = (6.838 \pm 0.072) \cdot 10^{-4} \text{ MeV}^{-1} , \quad (64)$$

$$Z = 1.71 \pm 0.18 . \quad (65)$$

The four coefficients c_{i,σ_N} and $c_{i,\rho}$, $i = 1, 2$, defined in Eqs. (31), (32), (36), and (37), are then:

$$c_{1,\sigma_N} = (-1.62 \pm 0.34) \cdot 10^4 \text{ MeV} , \quad (66)$$

$$c_{2,\sigma_N} = (-0.0151 \pm 0.0032) \text{ MeV}^{-1} , \quad (67)$$

$$c_{1,\rho} = -7.4 \pm 1.6 , \quad (68)$$

$$c_{2,\rho} = (4.13 \pm 0.93) \cdot 10^{-6} \text{ MeV}^{-2} . \quad (69)$$

Using these values, the low-energy coupling constants of the eLSM, Eqs. (40) – (43), are:

$$C_{1,eLSM} = -0.268 \pm 0.021 , \quad (70)$$

$$C_{2,eLSM} = (5.399 \pm 0.081) \cdot 10^{-5} \text{ MeV}^{-2} , \quad (71)$$

$$C_{3,eLSM} = (-9.30 \pm 0.59) \cdot 10^{-11} \text{ MeV}^{-4} + \left(-\frac{g_3}{4} + \frac{g_4}{4}\right) w^4 Z^4 , \quad (72)$$

$$C_{4,eLSM} = (9.45 \pm 0.59) \cdot 10^{-11} \text{ MeV}^{-4} + \frac{g_3}{2} w^4 Z^4 , \quad (73)$$

where the errors are calculated using the standard procedure associated to the χ^2 minimization described in Ref. [18]. The following comments are in order:

- 1) The constant $C_{1,eLSM}$ turns out to be in good agreement with the ChPT result. The eLSM value has, quite remarkably, an even smaller error than $C_{1,\chi PT}$. However, this does not mean that we can determine ℓ_3 to better precision than ChPT. Naively one would think that Eq. (5) allows us to express ℓ_3 as a function of M^2 , which, by Eq. (6), is linearly related to $C_{1,\chi PT}$. We could now replace $C_{1,\chi PT}$ with $C_{1,eLSM}$ and hope to obtain a smaller error for ℓ_3 than ChPT. This, however, does not work: we obtain $\ell_3 = (23 \pm 28) \cdot 10^{-3}$, i.e., although the value is consistent with the one quoted in Eq. (47) it has an error which is about one order of magnitude larger. The reason is that the mass difference $M_\pi^2 - M^2$ has a larger error which influences this way of determining ℓ_3 .

- 2) The quantity $C_{2,eLSM}$ is a few standard deviations off the NLO ChPT value. However, if we consider $C_{2,eLSM}(2f_\pi^2)$, with the value for f_π as given by the fit of Ref. [18], $f_\pi = (96.3 \pm 0.7)$ MeV, we obtain $C_{2,eLSM}(2f_\pi^2) = 1.00129 \pm 0.00012$, i.e., almost exactly equal to unity (although the error is about a factor of 10 smaller than the deviation from unity). It is interesting to list the five terms contributing to the right-hand side of Eq. (41) separately:

$$C_{2,eLSM}(2f_\pi^2) = 0.53775 + 2.93915 - 4.98863 + 2.45817 + 0.05484 = 1.00129. \quad (74)$$

From this we conclude that the result $C_{2,eLSM}(2f_\pi^2) \simeq 1$ is actually due to nontrivial cancellations.

- 3) The quantities $C_{3,eLSM}$ and $C_{4,eLSM}$ cannot be uniquely determined because the constants g_3 and g_4 were not determined in the fit of Ref. [18]. However, we can estimate their values, by replacing the left-hand sides of Eqs. (72) and (73) by the ChPT values (51) and (52) and then solving for g_3 and g_4 . We obtain

$$g_3 = -74 \pm 32, \quad (75)$$

$$g_4 = 6 \pm 52. \quad (76)$$

Although the errors are large, the values are of a natural order of magnitude. In turn, it means that the terms proportional to g_3 and g_4 are expected to affect $\pi\pi$ scattering.

V. CONCLUSIONS AND OUTLOOK

In this work, we have presented a low-energy study of the eLSM with two quark flavors. We have integrated out all heavy mesons in the functional integral representation of the vacuum-to-vacuum transition amplitude and kept only terms contributing at tree-level and up to fourth order in powers of derivatives of the pion field. In this way, we have obtained a low-energy effective action which contains only pions. We have mapped this effective action to that of ChPT by choosing a definite coset representative and expanding the latter to fourth order in powers of derivatives of the pion fields. This allowed us to compare the coefficients of the various terms, here termed low-energy coupling constants, in the low-energy effective action of the eLSM with the corresponding ones of ChPT.

The low-energy coupling constant $C_{1,\chi PT}$ is related to the LEC ℓ_3 , cf. Eqs. (5) and (6), while $C_{2,\chi PT} = 1/(2f_\pi^2)$. On the other hand $C_{1,eLSM}$ and $C_{2,eLSM}$ were determined from the fit of Ref. [18]. We found reasonable agreement between $C_{1,eLSM}$ and $C_{1,\chi PT}$, while the numerical values of $C_{2,eLSM}$ and $C_{2,\chi PT}$ differ by a few standard deviations. However, if we consider the dimensionless quantity $C_{2,eLSM}(2f_\pi^2)$, with f_π from the fit of Ref. [18], we obtain within about 0.13% the value $C_{2,eLSM}(2f_\pi^2) = 1$.

A direct comparison of the low-energy coupling constants $C_{3,eLSM}$ and $C_{4,eLSM}$ with the corresponding ones in ChPT was at present not possible because the coupling constants g_3 and g_4 have not yet been determined. In view of this we reverted the argument and obtained an estimate for these couplings constants by equating $C_{3,eLSM} \equiv C_{3,\chi PT}$ and $C_{4,eLSM} \equiv C_{4,\chi PT}$. We obtained values which are of a natural order of magnitude. It would be interesting to study $\pi\pi$ scattering within the eLSM to confirm the values obtained here.

In conclusion, we confirmed the validity of the eLSM as an effective hadronic model by showing that its low-energy limit correctly reproduces the low-energy coupling constants of ChPT to NLO. A necessary ingredient proved to be the inclusion of (axial-)vector degrees of freedom. In App. B 1 we corroborate this conclusion by studying a scenario without (axial-)vector mesons. Let us also repeat the main conclusion of Ref. [18], namely that the scalar quark-antiquark states lie above 1 GeV in mass and in particular that the chiral partner of the pion has to be identified with the $f_0(1370)$ resonance. In App. B 2 we study an alternative scenario where the scalar quark-antiquark states are identified with resonances below 1 GeV in mass. In this case, we show that the low-energy coupling constants of the eLSM disagree with those of ChPT.

At present, our conclusions hold at tree-level. Therefore, a mandatory future project is to compute loop corrections to the low-energy coupling constants for the eLSM in order to confirm that the eLSM has the same low-energy effective action as QCD.

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Appendix A: Lagrangians

1. ChPT

At LO, the ChPT Lagrangian is given by

$$\mathcal{L}_2 = \frac{f_\pi^2}{4} \text{Tr} \left\{ (D_\mu \mathcal{U})^\dagger D^\mu \mathcal{U} \right\} + \frac{f_\pi^2}{4} \text{Tr} \left\{ \chi^\dagger \mathcal{U} + \mathcal{U}^\dagger \chi \right\} , \quad (\text{A1})$$

where $D_\mu \mathcal{U} = \partial_\mu \mathcal{U} - i r_\mu \mathcal{U} + i l_\mu \mathcal{U}$, with external left- and right-handed vector fields l_μ , r_μ , respectively, and where $\chi = 2B_0(s + ip)$, with the LEC B_0 and external scalar and pseudoscalar fields s and p , respectively.

At NLO, the number of terms increases to ten. In trace notation, the respective Lagrangian is given by

$$\begin{aligned} \mathcal{L}_4 = & \frac{\ell_1}{4} \left(\text{Tr} \left\{ (D_\mu \mathcal{U})^\dagger D^\mu \mathcal{U} \right\} \right)^2 + \frac{\ell_2}{4} \text{Tr} \left\{ (D_\mu \mathcal{U})^\dagger D_\nu \mathcal{U} \right\} \text{Tr} \left\{ (D^\mu \mathcal{U})^\dagger D^\nu \mathcal{U} \right\} \\ & + \frac{h_1 - h_3 + \ell_3}{16} \left(\text{Tr} \left\{ \chi^\dagger \mathcal{U} + \mathcal{U}^\dagger \chi \right\} \right)^2 + \frac{\ell_4}{4} \text{Tr} \left\{ (D_\mu \mathcal{U})^\dagger D^\mu \chi + (D_\mu \chi)^\dagger D^\mu \mathcal{U} \right\} \\ & + \ell_5 \text{Tr} \left\{ f_{\mu\nu}^{(R)} \mathcal{U} f^{(L)\mu\nu} \mathcal{U}^\dagger \right\} - \left(\frac{\ell_5}{2} + 2h_2 \right) \text{Tr} \left\{ f_{\mu\nu}^{(L)} f^{(L)\mu\nu} + f_{\mu\nu}^{(R)} f^{(R)\mu\nu} \right\} \\ & + i \frac{\ell_6}{2} \text{Tr} \left\{ f_{\mu\nu}^{(R)} (D^\mu \mathcal{U}) (D^\nu \mathcal{U})^\dagger + f_{\mu\nu}^{(L)} (D^\mu \mathcal{U})^\dagger D^\nu \mathcal{U} \right\} + \frac{h_1 - h_3 - \ell_7}{16} \left(\text{Tr} \left\{ \chi^\dagger \mathcal{U} - \mathcal{U}^\dagger \chi \right\} \right)^2 \\ & + \frac{h_1 + h_3}{4} \text{Tr} \left\{ \chi^\dagger \chi \right\} - \frac{h_1 - h_3}{8} \text{Tr} \left\{ \chi \mathcal{U}^\dagger \chi \mathcal{U} + \mathcal{U} \chi^\dagger \mathcal{U} \chi \right\} , \end{aligned} \quad (\text{A2})$$

where $f_{\mu\nu}^{(L,R)}$ are the field-strength tensors of external left- and right-handed vector fields, and $\ell_1, \dots, \ell_7, h_1, h_2$, and h_3 are LECs (please do not confuse h_1, h_2, h_3 with the respective coupling constants of the eLSM).

2. Terms $\sim H\pi^2$ and $\sim H^2\pi^2$

We report here the Lagrangians containing interactions of the form $H\pi^2$ and $H^2\pi^2$ for the heavy fields:

$$\begin{aligned} \mathcal{L}_{\sigma_N \pi} = & \frac{1}{2} (\partial_\mu \sigma_N)^2 - \frac{1}{2} m_{\sigma_N}^2 \sigma_N^2 - \left(\lambda_1 + \frac{\lambda_2}{2} \right) \phi_N Z^2 \sigma_N \vec{\pi}^2 + g_1 w Z^2 (\partial^\mu \sigma_N) (\partial_\mu \vec{\pi}) \cdot \vec{\pi} \\ & + \left\{ \left[g_1^2 \phi_N + (h_1 + h_2 - h_3) \frac{\phi_N}{2} \right] w^2 Z^2 - g_1 w Z^2 \right\} \sigma_N (\partial_\mu \vec{\pi})^2 \\ & + \left[\frac{g_1^2}{2} + \frac{1}{4} (h_1 + h_2 - h_3) \right] w^2 Z^2 \sigma_N^2 (\partial_\mu \vec{\pi})^2 - \frac{1}{2} \left(\lambda_1 + \frac{\lambda_2}{2} \right) Z^2 \sigma_N^2 \vec{\pi}^2 , \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} \mathcal{L}_{a_0 \pi} = & \frac{1}{2} (\partial_\mu \vec{a}_0)^2 - \frac{1}{2} m_{a_0}^2 \vec{a}_0^2 + \frac{\lambda_2}{2} Z^2 (\vec{a}_0 \cdot \vec{\pi})^2 - \frac{1}{2} \left(\lambda_1 + \frac{3\lambda_2}{2} \right) Z^2 \vec{a}_0^2 \vec{\pi}^2 \\ & + \frac{1}{4} (h_1 + h_2 - h_3) w^2 Z^2 \vec{a}_0^2 (\partial_\mu \vec{\pi})^2 + \frac{g_1^2}{2} w^2 Z^2 (\vec{a}_0 \cdot \partial_\mu \vec{\pi})^2 + \frac{h_3}{2} w^2 Z^2 (\vec{a}_0 \times \partial_\mu \vec{\pi})^2 , \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} \mathcal{L}_{\eta_N \pi} = & \frac{1}{2} (\partial_\mu \eta_N)^2 - \frac{1}{2} m_{\eta_N}^2 \eta_N^2 + \left[\frac{g_1^2}{2} + \frac{1}{4} (h_1 + h_2 - h_3) \right] w^2 Z^4 (\partial_\mu \eta_N)^2 \vec{\pi}^2 \\ & + \left[\frac{g_1^2}{2} + \frac{1}{4} (h_1 + h_2 - h_3) \right] w^2 Z^4 \eta_N^2 (\partial_\mu \vec{\pi})^2 - \frac{1}{2} \left(\lambda_1 + \frac{3\lambda_2}{2} \right) Z^4 \eta_N^2 \vec{\pi}^2 \\ & + (2g_1^2 + h_2 - h_3) w^2 Z^4 \eta_N (\partial^\mu \eta_N) \vec{\pi} \cdot (\partial_\mu \vec{\pi}) + (g_3 + g_4) w^4 Z^4 (\partial^\mu \eta_N) (\partial^\nu \eta_N) (\partial_\mu \vec{\pi}) \cdot (\partial_\nu \vec{\pi}) \\ & + \left(\frac{g_3}{2} + \frac{g_4}{2} + \frac{g_5}{2} + g_6 \right) w^4 Z^4 (\partial_\mu \eta_N)^2 (\partial_\nu \vec{\pi})^2 , \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} \mathcal{L}_{\omega_N \pi} = & -\frac{1}{4} \omega_N^{\mu\nu} \omega_{N,\mu\nu} + \frac{1}{2} m_{\omega_N}^2 \omega_{N,\mu}^2 - \frac{\xi_{\omega_N}}{2} (\partial_\mu \omega_N^\mu)^2 + \frac{1}{4} (h_1 + h_2 + h_3) Z^2 \omega_{N,\mu}^2 \vec{\pi}^2 \\ & + \left(\frac{g_3}{2} + \frac{g_4}{2} + \frac{g_5}{2} + g_6 \right) w^2 Z^2 \omega_{N,\mu}^2 (\partial_\nu \vec{\pi})^2 + (g_3 + g_4) w^2 Z^2 \omega_N^\mu \omega_N^\nu (\partial_\mu \vec{\pi}) \cdot (\partial_\nu \vec{\pi}) , \end{aligned} \quad (\text{A6})$$

$$\begin{aligned}\mathcal{L}_{f_{1N}\pi} = & -\frac{1}{4}f_{1N}^{\mu\nu}f_{1N,\mu\nu} + \frac{1}{2}m_{f_{1N}}^2 f_{1N,\mu}^2 - \frac{\xi_{f_{1N}}}{2}(\partial_\mu f_{1N}^\mu)^2 + \left[\frac{g_1^2}{2} + \frac{1}{4}(h_1 + h_2 - h_3)\right] Z^2 f_{1N,\mu}^2 \vec{\pi}^2 \\ & + \left(\frac{g_3}{2} + \frac{g_4}{2} + \frac{g_5}{2} + g_6\right) w^2 Z^2 f_{1N,\mu}^2 (\partial_\nu \vec{\pi})^2 + (g_3 + g_4) w^2 Z^2 f_{1N}^\mu f_{1N}^\nu (\partial_\mu \vec{\pi}) \cdot (\partial_\nu \vec{\pi}) ,\end{aligned}\quad (\text{A7})$$

$$\begin{aligned}\mathcal{L}_{\rho\pi} = & -\frac{1}{4}\vec{\rho}^{\mu\nu} \cdot \vec{\rho}_{\mu\nu} + \frac{1}{2}m_\rho^2 \vec{\rho}_\mu^2 - \frac{\xi_\rho}{2}(\partial_\mu \vec{\rho}^\mu)^2 + g_2 w^2 Z^2 (\partial^\mu \vec{\rho}^\nu) \cdot (\partial_\nu \vec{\pi} \times \partial_\mu \vec{\pi}) \\ & + \left[(g_1^2 \phi_N - h_3 \phi_N) w Z^2 - g_1 Z^2\right] \vec{\rho}^\mu \cdot (\partial_\mu \vec{\pi} \times \vec{\pi}) + \frac{1}{2}(g_1^2 - h_3) Z^2 (\vec{\pi} \times \vec{\rho}_\mu)^2 + \frac{1}{4}(h_1 + h_2 + h_3) Z^2 \vec{\rho}_\mu^2 \vec{\pi}^2 \\ & + \left(-\frac{g_3}{2} + \frac{g_4}{2} + \frac{g_5}{2} + g_6\right) w^2 Z^2 \vec{\rho}_\mu^2 (\partial_\nu \vec{\pi})^2 + (-g_3 + g_4 - g_5 + 2g_6) w^2 Z^2 (\vec{\rho}^\mu \cdot \partial_\mu \vec{\pi}) (\vec{\rho}^\nu \cdot \partial_\nu \vec{\pi}) \\ & + g_3 w^2 Z^2 [(\vec{\rho}^\mu \cdot \vec{\rho}^\nu) (\partial_\mu \vec{\pi}) \cdot (\partial_\nu \vec{\pi}) + (\vec{\rho}^\mu \cdot \partial^\nu \vec{\pi}) (\vec{\rho}_\mu \cdot \partial_\nu \vec{\pi}) + (\vec{\rho}^\mu \cdot \partial^\nu \vec{\pi}) (\vec{\rho}_\nu \cdot \partial_\mu \vec{\pi})] ,\end{aligned}\quad (\text{A8})$$

$$\begin{aligned}\mathcal{L}_{a_1\pi} = & -\frac{1}{4}\vec{a}_1^{\mu\nu} \cdot \vec{a}_{1,\mu\nu} + \frac{1}{2}m_{a_1}^2 \vec{a}_{1,\mu}^2 - \frac{\xi_{a_1}}{2}(\partial_\mu \vec{a}_1^\mu)^2 + \frac{g_1^2}{2} Z^2 (\vec{a}_{1,\mu} \cdot \vec{\pi})^2 + \frac{1}{4}(h_1 + h_2 - h_3) Z^2 \vec{a}_{1,\mu}^2 \vec{\pi}^2 \\ & + \left(-\frac{g_3}{2} + \frac{g_4}{2} + \frac{g_5}{2} + g_6\right) w^2 Z^2 \vec{a}_{1,\mu}^2 (\partial_\nu \vec{\pi})^2 + \frac{h_3}{2} Z^2 (\vec{a}_{1,\mu} \times \vec{\pi})^2 \\ & + (-g_3 + g_4 + g_5 + 2g_6) w^2 Z^2 (\vec{a}_1^\mu \cdot \partial_\mu \vec{\pi}) (\vec{a}_1^\nu \cdot \partial_\nu \vec{\pi}) \\ & + g_3 w^2 Z^2 [(\vec{a}_1^\mu \cdot \partial^\nu \vec{\pi}) (\vec{a}_{1,\mu} \cdot \partial_\nu \vec{\pi}) + (\vec{a}_1^\mu \cdot \partial^\nu \vec{\pi}) (\vec{a}_{1,\nu} \cdot \partial_\mu \vec{\pi}) + (\vec{a}_1^\mu \cdot \vec{a}_1^\nu) (\partial_\mu \vec{\pi}) \cdot (\partial_\nu \vec{\pi})] .\end{aligned}\quad (\text{A9})$$

In the end, it is also useful to add a Stückelberg term for each of the (axial-)vector mesons:

$$\mathcal{L}_{ST} = -\frac{\xi_{\omega_N}}{2}(\partial_\mu \omega_N^\mu)^2 - \frac{\xi_{f_{1N}}}{2}(\partial_\mu f_{1N}^\mu)^2 - \frac{\xi_\rho}{2}(\partial_\mu \vec{\rho}^\mu)^2 - \frac{\xi_{a_1}}{2}(\partial_\mu \vec{a}_1^\mu)^2 . \quad (\text{A10})$$

Appendix B: Other scenarios

In this appendix we consider different scenarios. We first discuss the limiting case where vector mesons decouple (this is the case of the original LSM with only scalar and pseudoscalar states). Then, we describe the case where the scalar mesons are lighter than 1 GeV [which was not favored by the fit of Ref. [18]].

1. Results without vector mesons

The limit where (axial-)vector mesons decouple is realized by setting $g_i = 0$, $i = 1, \dots, 6$, $h_i = 0$, $i = 1, 2, 3$ in the Lagrangian (14). As a consequence, $Z = 1$, $w = 0$, and $\phi_N = f_\pi$. Moreover, $c_{2,\sigma_N} = c_{1,\rho} = c_{2,\rho} = 0$ and

$$c_{1,\sigma_N} = -\left(\lambda_1 + \frac{\lambda_2}{2}\right) f_\pi . \quad (\text{B1})$$

The corresponding low-energy coupling constants read in this limit:

$$C_{1,LSM} = -\frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{2}\right) + \frac{c_{1,\sigma_N}^2}{2m_{\sigma_N}^2}\left(1 - \frac{4m_\pi^4}{m_{\sigma_N}^4}\right) , \quad (\text{B2})$$

$$C_{2,LSM} = \frac{2c_{1,\sigma_N}^2}{m_{\sigma_N}^4}\left(1 + \frac{4m_\pi^2}{m_{\sigma_N}^2}\right) , \quad (\text{B3})$$

$$C_{3,LSM} = \frac{2c_{1,\sigma_N}^2}{m_{\sigma_N}^6} , \quad (\text{B4})$$

$$C_{4,LSM} = 0 . \quad (\text{B5})$$

The following comments are in order:

- 1) For $m_{\sigma_N} = 1362.7$ MeV, we obtain $C_{1,LSM} = -5.869 \pm 0.226$, which is more than an order of magnitude off the value in nature.

2) Upon replacing the coupling constants λ_1 and λ_2 by the masses of the σ_N -meson and the pion, we obtain

$$c_{1,\sigma_N}^2 = \left(\lambda_1 + \frac{\lambda_2}{2} \right)^2 f_\pi^2 = \frac{(m_{\sigma_N}^2 - m_\pi^2)^2}{4f_\pi^2} \quad (\text{B6})$$

and

$$C_{2,LSM}(2f_\pi^2) = \left(1 - \frac{m_\pi^2}{m_{\sigma_N}^2} \right)^2 \left(1 + \frac{4m_\pi^2}{m_{\sigma_N}^2} \right). \quad (\text{B7})$$

Only for $m_{\sigma_N} \rightarrow \infty$ we recover the NLO ChPT result

$$C_{2,LSM} = \frac{1}{2f_\pi^2} \equiv C_{2,\chi PT}. \quad (\text{B8})$$

This result is expected from a mathematical point of view because this limit corresponds to the nonlinear sigma model. For any finite value of m_{σ_N} the low-energy coupling constant $C_{2,LSM}$ in principle deviates from $C_{2,\chi PT}$, but in reality, when $m_{\sigma_N} \gtrsim 2.3$ GeV, they still agree within errors. Including (axial-)vector mesons, m_{σ_N} may also be smaller.

- 3) $C_{3,LSM}$ receives a positive contribution from the (pseudo)scalar sector. However, its true value is negative, cf. Eq. (51). The (pseudo)scalar sector alone is not sufficient to obtain agreement with data.
- 4) $C_{4,LSM}$ vanishes, contrary to the value obtained in nature, cf. Eq. (52). This quantity depends entirely on the presence of vector mesons. Without them, agreement with data cannot be obtained.

2. Light scalars as $\bar{q}q$ states

An interesting and important issue in the field of hadron spectroscopy is to clarify which scalar-isoscalar state is the chiral partner of the pion. As already shown in Refs. [16–19], the best fit of the parameters of the eLSM to hadron masses and decay widths implies that the chiral partner lies above 1 GeV in mass and should be identified with the state $f_0(1370)$. Indeed, as discussed by many authors [see e.g. Refs. [26–28] and the recent review [29]], $f_0(500)$ should be regarded as a four-quark state.

Yet, it is still interesting to consider the – by now unfavored – scenario where $f_0(500)$ is (predominantly) a quark-antiquark state (and thus the chiral partner of the pion). The scenario where the scalar states lie below 1 GeV corresponds to the third best fit in Ref. [18] with a $\chi^2/\text{d.o.f.} = 11.8$. One obtains $C_{1,eLSM} = -0.4388 \pm 0.0002$ and $C_{2,eLSM} = (9.093 \pm 0.003) \cdot 10^{-5} \text{ MeV}^{-2}$. Both values are almost a factor of two larger than in nature, but due to the large error in Eq. (49), the value of $C_{1,eLSM}$ is still acceptable. However, that of $C_{2,eLSM}$ is further off unity than for the scenario with the heavy scalars: $C_{2,eLSM}(2f_\pi^2) \simeq 1.030672 \pm 0.000018$. Thus, we confirm that the assignment of light scalars to ordinary quark-antiquark mesons is less favored. This is in agreement with the recent findings of Ref. [30], in which the light scalar states $a_0(980)$ and $K_0^*(800)$ could be determined – in the context of Lagrangians derived from the eLSM – as dynamically generated companion poles of quarkonia states above 1 GeV in mass, namely $a_0(1450)$ and $K_0^*(1430)$.

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